Mathematical Induction 2

1. Prove by Mathematical induction,

   $P(n): \frac{1}{2n} \leq \frac{1.3.5...(2n-1)}{2.4.6...(2n)}$ where n $\in$**N.**

   P(n) : $\frac{1}{2n} \leq \frac{1.3.5...(2n-1)}{2.4.6...(2n)}$

   For P(1), L.H.S.=$\frac{1}{2\times 1} = \frac{1}{2} \leq \frac{1}{2}$ =R.H.S., $\therefore$ P(1) is true.

   Assume P(k) is true for some k$\in$ N, that is,$\frac{1}{2k} \leq \frac{1.3.5...(2k-1)}{2.4.6...(2k)}$.........(1)

   For P(k + 1), $\frac{1}{2(k+1)} = \frac{1}{2k} \times \frac{2k}{2(k+1)} \leq \frac{1.3.5...(2k-1)}{2.4.6...(2k)} \times \frac{2k}{2(k+1)}$, by (1)

   $= \frac{1.3.5...(2k-1)}{2.4.6...(2k)} \times \left[\frac{(2k+1)}{2(k+1)} \times \frac{2k}{(2k+1)}\right] = \left[\frac{1.3.5...(2k-1)}{2.4.6...(2k)} \times \frac{(2k+1)}{2(k+1)}\right] \times \frac{2k}{(2k+1)}$

   $= \frac{1.3.5...(2k-1)(2k+1)}{2.4.6...(2k)[2(k+1)]} \times \frac{2k}{2k+1} \leq \frac{1.3.5...(2k-1)(2k+1)}{2.4.6...(2k)[2(k+1)]} \times 1 = \frac{1.3.5...(2k-1)(2k+1)}{2.4.6...(2k)[2(k+1)]}$

   $\therefore$ P(k + 1) is true.

   By the Principle of Mathematical Induction, P(n) is true $\forall$ n $\in$ N.

2. Prove by mathematical induction:

   $1^2 + 2^2 + 3^2 + \cdots + (2n)^2 = \frac{[n\ (2n+1)(4n+1)]}{3}$ for the first $2n$ positive integers.

   Let $P(n): 1^2 + 2^2 + 3^2 + \cdots + (2n)^2 = \frac{n\ (2n+1)(4n+1)}{3}$

   For $P(1): 1^2 + 2^2 = 5 = \frac{1\ (2(1)+1)(4(1)+1)}{3}$. $\therefore P(1)$ is true.

   Assume $P(k)$ is true for some $k \in \boldsymbol{N}$, that is

   $1^2 + 2^2 + 3^2 + \cdots + (2k)^2 = \frac{k(2k+1)(4k+1)}{3}$ ... (1)

   For $P(k+1)$:

   $1^2 + 2^2 + 3^2 + \cdots + (2k)^2 + (2k+1)^2 + \left(2(k+1)\right)^2$

   $= \frac{k(2k+1)(4k+1)}{3} + (2k+1)^2 + \left(2(k+1)\right)^2$, by (1).

   $= \frac{2k+1}{3}[k(4k+1) + 3(2k+1)] + 4(k+1)^2$

   $= \frac{2k+1}{3}[4k^2 + 7k + 3] + 4(k+1)^2 = \frac{1}{3}(2k+1)(k+1)(4k+3) + 4(k+1)^2$

   $= \frac{1}{3}(k+1)[(2k+1)(4k+3) + 12(k+1)] = \frac{1}{3}(k+1)[8k^2 + 22k + 15]$

$= \frac{1}{3}(k+1)(2k+3)(4k+5) = \frac{(k+1)[2(k+1)+1][4(k+1)+1]}{3}$

$\therefore P(k+1)$ is true.

By the Principle of mathematical induction, $P(n)$ is true for all $n \in \boldsymbol{N}$.

3. Prove by mathematical induction: $6|(n^3 - n)$ for all natural values of n.

Let $P(n)$: (1) $n^2 + n = 2a_n$

(2) $n^3 - n = 6b_n$ where $a_n, b_n \in \boldsymbol{Z}$.

For $P(1)$: (1) $1^2 + 1 = 2 = 2 \times 1$, $a_1 = 1$

(2) $1^3 - 1 = 0 = 6 \times 0$, $b_1 = 0$.

$\therefore P(1)$ is true.

Assume $P(k)$ is true for some $k \in \boldsymbol{N}$, that is

$k^2 + k = 2a_k \ldots (i)$

$k^3 - k = 6b_k \ldots (ii)$ where $a_k, b_k \in \boldsymbol{Z}$.

For $P(k+1)$:

(1) $(k+1)^2 + (k+1) = (k^2 + 2k + 1) + (k+1) = (k^2 + k) + 2(k+1)$

$= 2a_k + 2(k+1)$, by (i)

$= 2(a_k + k + 1) = 2a_{k+1}$

(2) $(k+1)^3 - (k+1) = (k^3 + 3k^2 + 3k + 1) - (k+1)$

$= (k^3 - k) + 3(k^2 + k)$

$= 6b_k + 3(2a_k) = 6(b_k + a_k) = 6b_{k+1}$

$\therefore P(k+1)$ is true.

By the Principle of mathematical induction, $P(n)$ is true for all $n \in \boldsymbol{N}$.

Therefore $6|(n^3 - n)$ for all natural values of n.

4. Prove that $(9^n - 8n - 1)$ is divisible by 8 for all non-negative integers, n .

Let $P(n): 9^n - 8n - 1 = 8a_n$ where $a_n \in \boldsymbol{Z}, n \in \boldsymbol{N} \cup \{0\}$.

For $P(0): 9^0 - 8(0) - 1 = -8 = 8a_0$, $a_0 = -1$

Assume $P(k)$ is true for some $k \in N$

We have $9^k - 8k - 1 = 8a_k \ldots (1)$

For $P(k+1)$,

$9^{k+1} - 8(k+1) - 1$

$= 9(9^k - 8k - 1) + 64k$

$= 9(8a_k) + 8(8k)$ by (1)

$= 8(9a_k + 8k)$,

$= 8a_{k+1}$

$\therefore P(k+1)$ is true.

By the Principle of mathematical induction, $P(n)$ is true for all $n \in \boldsymbol{N} \cup \{0\}$.

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